Amenability of horocyclic products of percolation trees

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Abstract: For horocyclic products of percolation subtrees of regular trees, we show almost sure amenability. Under a symmetry condition concerning the growth of the two percolation trees, we show the existence of an *increasing* Følner sequence (which we call *strong* amenability).

Keywords: (strong) amenability, anchored expansion, isoperimetric constant, Diestel-Leader graphs, percolation

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1 Introduction

1.1 Amenability and strong amenability of Diestel-Leader graphs

The family of graphs called *Diestel-Leader graphs* (=: DL graphs), which are of exponential growth, have the special property that they include an amenable subfamily, and are otherwise non-amenable and even non-unimodular (see [4]). They have been designed to answer a question posed by Woess [22] about the existence of quasi-transitive graphs which are not quasi-isometric to a Cayley graph of a group. An answer has been given recently in [9]. These graphs are certain ('horocyclic') products (for the definition, see section 1.4) of homogeneous trees, which, if taken for two trees of equal degree, turn out to be the Cayley graphs of the lamplighter group on the integers [2, 5]. As shown by Kaimanovich and Vershik [13], the speed of the simple random walk on the Cayley graph of the lamplighter group on $\mathbb{Z}^d$ is zero, iff $d \in \{1,\ 2\}$. Therefore, the speed of simple random walk on DL-graphs for trees of equal degree is zero.

Virag[24] has proven that the positivity of the speed of a simple random walk on an infinite graph is implied by the positivity of the anchored isoperimetric constant (**anchored expansion**). It can be shown, that for symmetric Diestel-Leader graphs (i.e. trees of equal degree), there is a Følner sequence [2], implying amenability. Even the anchored isoperimetric constant vanishes, a situation which is called **strong amenability** in [11]. From a paper by Chen, Peres and Pete this follows for Bernoulli percolation on symmetric Diestel-Leader graphs [5]. Our results, however, refer to a modification of Bernoulli percolation: similar to Diestel-Leader graphs, two trees are 'coupled' by requiring their Busemann functions [26] to add up to zero. However, the tees involed are independent bond-percolation subtrees of regular trees. In spite of there possibly being no vertices of degree two, leading to large subgraphs consisting only of finite chains ('stretchings'), there is amenability, almost surely. Moreover, when the process is chosen in a symmetric way on the factors of the horocyclic product, we show a.s. *strong* amenability (Theorem 2.1).

It has been shown in [5] (see also [17]) that the super-critical percolation cluster of an invariant percolation containing a pre-assigned 'root' is a.s. weakly non-amenable in the case of an underlying non-amenable graph and the percolation being Bernoulli if the retention parameter p is sufficiently close to one. In [11], it has been proven that this cluster remains a.s. strongly amenable under the 'random pertubation' given by invariant percolation, if the underlying graph is amenable and transitive. Our results complement these findings by showing amenability of a certain percolation model different from Bernoulli percolation. This percolation process is defined by coupling two independent subtrees which result from independent bond-percolation. The coupling is done according to the rules of constructing Diestel-Leader graphs ([27], see also [26], chapter 12.18), which we will call the horocyclic product. We prove a.s. strong amenability at a point p_o in the range of parameters at which there is equal growth of the random trees involved.

1.2 Notation

The letter $G = \langle V, E\rangle$ will be denoted fore the deterministic 'underlying' transitive (Diestel-Leader) graph, on the edges of which a percolation process will be defined. '$H(\bar{\omega})$' (or just H) will be reserved for the random ('percolative') subgraphs $\langle \bar{V}, \bar{E}\rangle$ of G. Since we will deal with a product probability space $\bar{\Omega} = \Omega' \times \Omega$, its elements will be called $\bar{\omega} = \langle \omega', \omega\rangle$, throughout. Edges $\bar{e} \in \bar{E}$ will be undirected and denoted by subsets of the vertices: $\bar{e} = \{\bar{k}, \bar{l}\} \subset V$. There are no loops, such that every $\bar{e} \in E$ has two elements. The set $C_{\bar{o}} \subset \bar{V}$ will be the connected component of H containing a pre-assigned root $\bar{o} \in V$. We will focus on bond percolation graphs, such that $V = \bar{V}$ and H is a so called partial graph of G.

1.3 Products of trees with a fixed end

We recall the definitions concerning *trees with a fixed end*. For the following definitions, we refer to [4] and [26], for a more detailed discussion. A ray is an infinite sequence of successive neighboured vertices without repetitions. In a homogeneous tree $\mathcal{T}_M$ of degree $M = q + 1 \geq 3$, denote by $\partial\mathcal{T}_M$ its *boundary*, which is the union of all *ends*. An end is an equivalence class of *rays*, where two rays are equivalent if both have infinitely many vertices in common with a third. In particular, for trees, this means that the traces of two rays of the same end differ only by finitely many vertices.

After having chosen a *root*, denoted by 'o', and an element γ of $\partial\mathcal{T}_{q+1}$, it is possible to define the *Busemann function* $\mathbf{h}(x) := d(x, c_\gamma) - d(c_\gamma, o)$, where $x \curlywedge \gamma$ is the *confluent*, the last common vertex of the two geodesic rays between γ and x and γ and o, and $d(x, y)$ is the length of the geodesic ray between x and y. $\mathbf{h}(x)$ is the index of the 'level' of the vertex x on the directed tree with fixed end γ (see [14] , where the term pointed tree is used). In Diestel-Leader graphs, this level hierarchy is used to construct a product of trees (see [26], chapter 12.18). If this product involves two homogeneous trees of equal degree as its factors, the automorphism group of the resulting graph is the Cayley graph of an amenable group [27]. This results from the group being a closed subgroup of the Cartesian product of the amenable automorphism groups of the two involved trees. On the other hand, if the trees are homogeneous of *different* degree, the graph is not even unimodular ([26], chap. 12.18).

1.4 Horocyclic products

Let $\mathcal{T}' = \langle V(\mathcal{T}'), E(\mathcal{T}')\rangle$, and $\mathcal{T} = \langle V(\mathcal{T}), E(\mathcal{T})\rangle$ be two homogeneous trees of degree $\alpha'+1$ and $\alpha+1$ with fixed roots o' and o, and fixed ends γ' and γ, respectively. Due to the fixed end and fixed root, the vertex $k \in V(\mathcal{T})$ has a Busemann-function (=:level-coordinate) $\mathbf{h}(k)$, likewise for $k' \in V(\mathcal{T}')$: $\mathbf{h}'(k')$. Let $DL_{\alpha',\alpha} = \langle V, E\rangle$ be the graph with

$$V \quad = \quad \{\ \langle k', k\rangle \in V(\mathcal{T}') \times V(\mathcal{T}) \quad | \quad \mathbf{h}(k') = -\mathbf{h}(k)\ \}, \tag{1}$$

while the edge-set E is inherited by the edge-sets $E(\mathcal{T}')$ and $E(\mathcal{T})$:

$$E = \{ \ \{\langle k', k\rangle, \ \langle l', l\rangle\} \subset V \mid k' \sim l', k \sim l\}, \tag{2}$$

with $\sim$ meaning neighbours in $\mathcal{T}'$ and $\mathcal{T}$. The graphs $DL_{\alpha',\alpha}$ with $\alpha, \alpha' \in \{1, 2, 3, 4, ...\}$ are the Diestel Leader graphs. Since it is a subgraph of the product of $\mathcal{T}'$ and $\mathcal{T}$, we denote this *horocyclic product* by

$$\mathcal{T}' \circ \mathcal{T} := G := DL_{\alpha',\alpha}. \tag{3}$$

Let $V_h \subset V$ be the vertex set of the finite, connected induced subgraph $G^{(h)}$ of G, which contains all vertices $\bar{v} := \langle v', v\rangle$ with

$$|\mathbf{h}(v')| \leq h, \quad \text{and} \quad |\mathbf{h}(v)| \leq h,$$

for some given $h \in \mathbb{N}$ (see Fig. 1).

To conclude the introductory section, we present a lemma about horocyclic products, which will be used in the proofs of the main results. For two graphs G_1 and G_2 with disjoint vertex sets, we call $G_1 + G_2 := \langle V(G_1) \cup V(G_2), E(G_1) \cup E(G_2)\rangle$ the graph consisting of the connected components given by those of G_1 and G_2 without there being any additional connection (edge) between any of them (*disjoint union*, see [25]). By $G_1 \cup_{E'} G_2$ we mean the graph $\langle V(G_1) \cup V(G_2), E(G_1) \cup E(G_2) \cup E'\rangle$, such that if each of G_1 and G_2 is connected and $e = \{k, l\}$ with $k \in V(G_1), l \in V(G_2)$, then $G_1 \cup_{\{e\}} G_2$ is connected. We say E' **connects** G_1 **and** G_2.

Lemma 1.1. *Let T', T be two infinite trees, for each of which a Busemann function has been defined. Let T_1, T_2 subtrees of T with disjoint vertex sets, then $T' \circ (T_1 + T_2) = T' \circ T_1 + T' \circ T_2$, and this graph is disconnected. If $E' \subset E(T)$ connects T_1 and T_2, then $T' \circ (T_1 \cup_{E'} T_2)$ is connected.*

Proof: Let $k \in V(T_1)$, $l \in V(T_2)$, then $k, l \in V(T_1 + T_2)$ with no path in $T_1 + T_2$ connecting k and l. Since any path in $T' \circ (T_1 + T_2)$ connecting $\langle k', k\rangle$ with $\langle l', l\rangle$ for some vertices $k', l' \in V(T_1)$ is a graph of the form $P' \circ P$ with P' a path in T' (connecting k' and l', and P a path in T connecting k and l, $T' \circ (T_1 + T_2)$ is not connected, since such a path P does not exist. On the other hand, given two vertices $\langle k', k_1\rangle$ and $\langle l', k_2\rangle$ with $k_1, k_2 \in V(T_1)$, two paths can be found, $P' \leq T'$, connecting k' and l', and $P \leq T_1$, connecting k_1, and k_2, since T' and T_1 are connected. Then the graph $P' \circ P$ is a connected subgraph of $T' \circ T_1$, and since all edges are distinct, a connecting path itself, namely, connecting $\langle k', k_1\rangle$ and $\langle l', k_2\rangle$. The same holds for $T' \circ T_2$. Since every vertex in $T' \circ (T_1 + T_2)$ is either in $V(T' \circ T_1)$ or $V(T' \circ T_2)$, the graph $T' \circ (T_1 + T_2)$ consists of exactly these two connected components. Finally, when E' connects T_1 and T_2, for every two vertices $k, l \in V(T_1 \cup_E T_2)$, there is a path $P \leq T_1 \cup_E T_2$ connecting them, such that any two vertices in $V(T' \circ (T_1 \cup_E T_2))$ can be connected by a path of the form $P' \circ P$, where P' is a connecting path of k' and l' in T'. $\square$

2 Strong amenability of random horocyclic products

Definition: For any finite subgraph $H_f = \langle V_f, E_f \rangle$ of $G = \langle V, E \rangle$ of order $|V_f|$, define the **isoperimetric ratio $I_G(H_f)$ of H_f in G** by

$$I_G(H_f) \;:=\; \frac{|\partial_G V_f|}{|V_f|} \tag{4}$$

where $|\cdot|$ is cardinality, and $\partial_G V_f = \{k \in V \setminus V_f \;\;:\;\; \{k,l\} \in E, l \in V_f \;\}$ the (outer vertex-) boundary of V_f in G. The **anchored isoperimetric constant** [23][24] (in [24], $|\cdot|$ denotes the volume= sum of weights of a subgraph) is given by

$$I := \liminf_n \{\; I_G(H) \mid H \text{ is an order-}n \text{ connected subgraph of } G \text{ containing vertex } \bar{o} \;\}. \tag{5}$$

Note, that a slightly different definition of I can be given, which depends on the choice of the root (compare with the appendix of [11]). However, positivity of either constant implies the positivity of the other. We say, that G is **strongly amenable**, if I is zero. Otherwise, G is **weakly non-amenable**, or, that it has **anchored expansion**[24].

Recalling the definition of an amenable graph, which is given if the **isoperimetric constant** $I_o(G) = \inf\{I_G(H) | H \leq G, H \text{ finite}\}$ is zero, it is clear that an infinite graph with a vanishing *anchored* isoperimetric constant is amenable.

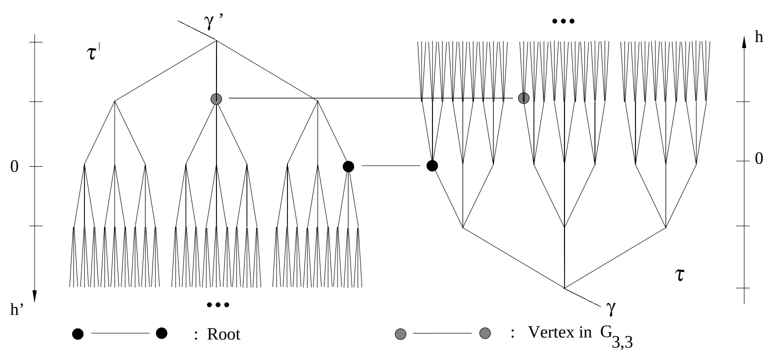

Figure 1: Subgraph $G^{(h)}$ of $G = \langle V, E \rangle = DL_{\alpha',\alpha}$ with $h = 2, \alpha = \alpha' = 3$ (The horizontal bars and the upside-down representation of one of the two trees refer to the condition of definition (1), [27]. An edge are two horizontal bars with vertices which are neighbours in the trees.)

Now, we consider a specific bond-percolation μ on the product sigma-algebra of $\Omega = 2^{E(\mathcal{T})}$ and likewise a percolation μ' on the product sigma-algebra of $\Omega' = 2^{E(\mathcal{T}')}$. Among the edges of G, choose a 'set of **remanent edges** $E_r \subset E$ and call $E_p := E \setminus E_r$ the 'set of **percolative edges**', in the following way. For a realization $\bar{\omega} = \langle \omega', \omega \rangle \in \Omega' \times \Omega$ in the product-probability space, let $H(\bar{\omega}) = \langle V, \bar{E}(\bar{\omega}) \rangle$ be the partial graph of G, given by

$$\bar{E}(\bar{\omega}) \quad := \quad \{ \ \{\langle k',k\rangle, \langle l',l\rangle\} \in E_p \ | \ \omega'(\{k',l'\}) = 1, \ \omega(\{k,l\}) = 1 \ \} \ \cup \ E_r.$$

$H(\bar{\omega})$, for $\bar{\omega} \in \Omega' \times \Omega$, is a bond-percolative subgraph of the Diestel-Leader graph G. At first, to increase clarity, we will be interested in the concrete example of $\alpha = \alpha' = 3$, and in the special Bernoulli percolation which allows only taking away 'the third edges'. For this purpose, for all vertices $\bar{k} = \langle k', k\rangle \in V$, we **mark** exactly one of the three edges of $\mathcal{T}$ (and $\mathcal{T}'$) incident to the vertices k (and k') pointing away from the fixed end γ (and γ') to be the set of pairs of edges (percolative edges E_p) which contain at least one marked edge of either tree. Let E_r (remanent edges) be the complement of E_p with respect to E.

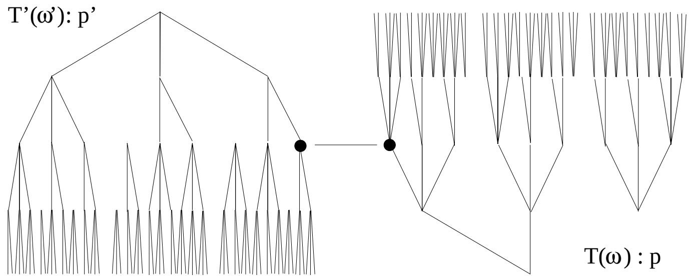

Figure 2: Percolative subgraph $H^{(2)}(\bar{\omega})$ of $G^{(2)}$: $\alpha_o' = \alpha_o = 2$, and $\alpha' = \alpha = 3$

In other words, for an edge $\bar{e} = \{\langle k',k\rangle, \langle l',l\rangle\} \in E$ to be a remanent edge ($\in E_r$), neither of the two edges $\{k',l'\} \in E(\mathcal{T}')$ and $\{k,l\} \in E(\mathcal{T})$ may be marked. The edges in E_r are not subject to the removal of the percolation. Let $H^{(h)}(\bar{\omega})$ be the subgraph of $H(\bar{\omega})$ with vertices $\bar{v} = \langle v', v\rangle$ fulfilling $|\mathbf{h}'(v')| \leq h, |\mathbf{h}(v)| \leq h$, then Figure 2 shows a typical realization of $H(\bar{\omega})$, when the unmarked edges consist of the first two children, and 'only some of the third edges are removed.'

We define the percolation measure $\bar{\mu}_{\bar{e}}$, with $\bar{e} \in E$ by the following: Let $E_m(\mathcal{T})$ (respectively, $E_m(\mathcal{T}')$) be the **marked edges** of $\mathcal{T}$ (respectively, $\mathcal{T}'$), and $E_u(\mathcal{T})$ (resp. $E_u(\mathcal{T}')$) be the **unmarked** ones: choose $E_u(\mathcal{T})$ by enumerating all edges incident to any vertex and pointing away from the end γ. Then call each of them part of $E_u(\mathcal{T})$, if the number labeling it is smaller or equal to a fixed integer $\alpha_o \in \{1, ..., \alpha\}$. Do the same to define $E_u(\mathcal{T}')$, with the integer α_o' as the number of 'unmarked children'. Let

$$\mu_e(\omega) = \chi_{E_u(\mathcal{T})}(e) \ + \ \chi_{E_m(\mathcal{T})}(e) \ \left(p \cdot \chi_{\{\omega(e)=1\}}(\omega) \ + \ (1-p) \cdot \chi_{\{\omega(e)=0\}}(\omega) \right), \tag{6}$$

with some $p \in (0,1)$, and $\mu'_{e'}$, with $e' \in E(T'), p' \in (0,1)$ by

$$\mu_{e'}(\omega') = \chi_{E_u(\mathcal{T}')}(e') \ + \ \chi_{E_m(\mathcal{T}')}(e') \ \left(p' \cdot \chi_{\{\omega'(e')=1\}}(\omega') \ + \ (1-p') \cdot \chi_{\{\omega'(e')=0\}}(\omega') \right). \tag{7}$$

Let $\bar{\mu} : \bar{e} = \langle e', e\rangle \in E(\mathcal{T}')\times E(\mathcal{T}) \mapsto \mu_{e'}\otimes\mu_e$ be the product measure on the product sigma-algebra of $\Omega'\times\Omega$. Figure 2 shows a subgraph of a realization with $\alpha'_o = \alpha_o = 2,\ \alpha = \alpha' = 3$. Note that we have for $\bar{e} = e' \circ e$ that $\bar{\mu}(\bar{e}\text{ open}) = \mu'_{e'}(e'\text{ open})\mu_e(e\text{ open})$. However, $\bar{\mu}$ is not an independent percolation measure, since e.g. the events that the two different edges $e' \circ e_1$ and $e' \circ e_2$ be open are positively correlated.

Then, for $\bar{\omega} \in \Omega' \times \Omega$, and $p', p \in [0, 1]$, call $C_{\bar{o}}(\bar{\omega})$ the connected component of $H(\bar{\omega})$, containing a preassigned root $\bar{o} = \langle o', o\rangle$. We say: $C_{\bar{o}}(\bar{\omega})$ is the connected component containing the root o and the ends γ', γ of the horocyclic product of two percolative subtrees $T' = T'(\omega')$ and $T = T(\omega)$ of retention parameter p', and p, respectively.

Remarks: i.) $T(\omega')' \circ T(\omega)$ restricted to $C_{\bar{o}}(\bar{\omega})$ is the horocyclic product of two rooted trees (with fixed ends) drawn from the **augmented Galton Watson measure** [1], with offspring distribution $\{p_k\}$ concentrated on $I := \{\alpha_o, ..., \alpha\}$, and being of binomial type

$$p_k \;=\; \binom{\alpha - \alpha_o}{k - \alpha_o} p^{k-\alpha_o}(1-p)^{\alpha-k+\alpha_o}, \quad \text{where } k \in I. \tag{8}$$

ii.) If $\alpha_o > 0$ or $\alpha'_o > 0$, the percolation $\bar{\mu}$ is *not* an invariant percolation: any vertex-transitive subgroup of the automorphism group $\mathrm{Aut}(G)$ of G must contain the operation which exchanges some vertex $\bar{k} = \langle k', k\rangle \in V$ with a vertex $\bar{l} = \langle l', l\rangle \in V$ at the same horocycle $(\mathbf{h}(k) = \mathbf{h}(l), \mathbf{h}'(k') = \mathbf{h}'(l'))$, where k is connected to its predecessor (parent) in $\mathcal{T}$ by a marked edge ($\in E_m(\mathcal{T})$) and l is connected to its parent by an unmarked edge ($\in E_u(\mathcal{T})$). Exchanging these vertices does not leave the measure $\bar{\mu}_{\bar{e}}$ invariant. In particular, an exchange of these two edges may lead to $C_{\bar{o}}(\bar{\omega})$ being disconnected from one of its fixed ends. On the other hand, if $\alpha_o = 0, \alpha'_o = 0$, the model corresponds to an invariant bond-percolation with retention parameter $p' \cdot p$ (not Bernoulli!). Equivalently, if $\alpha'_o = \alpha'$, $T'_{\alpha'_o,\alpha'}(\omega')$ is deterministic. If under these circumstances $\alpha_o = 0$, the model is also an invariant percolation, however with retention parameter p.

Theorem 2.1. *Let $p' = p \in [0, 1]$. Furthermore, let α'_o, α', and α_o, α be the minimum and maximum number of offspring at each site of $T'(\omega') = T_{\alpha'_o,\alpha'}(\omega')$ and $T(\omega) = T_{\alpha_o,\alpha}(\omega)$, respectively. Let $\alpha'_o, \alpha_o \geq 1$, and*

$$\alpha'_o \;+\; p'(\alpha' \;-\; \alpha'_o) \;\;=\;\; \alpha_o \;+\; p(\alpha \;-\alpha_o). \tag{9}$$

Then the restriction of the horocyclic product $H(\bar{\omega}) = T_{\alpha'_o,\alpha'} \circ T_{\alpha_o,\alpha}$ to the connected component containing the root $\bar{o}$, has $\bar{\mu}$-almost surely an anchored isoperimetric constant equal to zero, $\bar{\mu}$-almost surely, i.e. $H(\bar{\omega})|C_{\bar{o}}(\bar{\omega})$ is $\bar{\mu}$-a.s. strongly amenable.

Remark: iii.) The strong amenability ($\Leftrightarrow$ vanishing anchored isoperimetric constant [11]) of the Diestel-Leader graphs $DL_{\alpha,\alpha}$ is well known: e.g., it follows from the fact that the speed of the simple random walk is zero [13], together with the conclusion of [24], that anchored expansion implies positive speed. $DL_{\alpha'_o,\alpha_o}$ and $DL_{\alpha',\alpha}$ result as extremal cases

$\bar{\mu}$-a.s. if $p' = p = 0$ or $p' = p = 1$, respectively. The theorem says, that amenability is stable under the 'random pertubation' given by the specific construction of the percolation process, above, if the equal-growth condition (9) is met.

Proof: Let the root $\bar{o} := \langle o', o \rangle$ have the level coordinate 0 (see Fig.1). Let $C_o(\bar{\omega}) =: T'(\omega') \circ T(\omega)$, i.e. call $T'(\omega')$ and $T(\omega)$ the two random subtrees of $\mathcal{T}'$ and $\mathcal{T}$ with roots o' and o, respectively (see Fig. 2). Let $X_j^{(h)}(\omega)$ be the number of leaves (at level j) of the finite subtree $T_j^{(h)}(\omega)$ of $T(\omega)$, rooted at $-h$ with height $h+j$. Likewise, call $T_j^{\prime(h)}(\omega')$ the subtree of $T'(\omega')$ rooted at $+h$ with depth $h-j$ ($j \in \{-h, ..., h\}$) and $X_j^{\prime(h)}(\omega')$ its leaves, also located on level j.

If we find a Følner sequence, i.e. a sequence of finite, connected subgraphs of $T'(\omega') \circ T(\omega)$ restricted to the connected component $C_{\bar{o}}(\bar{\omega})$ containing the root, with an isoperimetric ratio converging to zero, then this graph (denoted by $T'(\omega') \circ T(\omega)|C_o(\bar{\omega})$) is strongly amenable (i.e. has vanishing anchored isoperimetric constant). In particular, this is true if the finite subgraphs given by $T_j^{\prime(h)}(\omega') \circ T_j^{(h)}(\omega)$ have an isoperimetric ratio $I_h(\bar{\omega})$ as subgraphs of $C_o(\bar{\omega})$, converging to zero, $\bar{\mu}$-almost surely, as $h \to \infty$. It is clear, that

$$I_h(\bar{\omega}) := \frac{X_h^{,(h)}(\omega') + X_h^{(h)}(\omega)}{\sum\limits_{j=-h}^{h} X_{-j}^{,(h)}(\omega') X_j^{(h)}(\omega)}. \tag{10}$$

We show $I_h(\bar{\omega}) \to 0$, $\bar{\mu}$-almost surely, to prove the theorem.

We note, there is a formular for $X_j^{(h)}$:

Lemma 2.2. *Let $e(k_1, ..., k_l)$ be the edge in the subtree of $\mathcal{T}$ rooted at $-h$ between the level $l-1$ and l, which is uniquely determined by the l-tuple of numbers $k_i \in \{1, 2, ..., \alpha\}$, with $i \in \{1, ..., l\}$ in an obvious way: among the α choices of children k_j is chosen on a path from the root at the jth step. Then*

$$X_j^{(h)}(\omega) = \sum_{k_1=1}^{\alpha} \cdots \sum_{k_{h+j}=1}^{\alpha} \prod_{l=1}^{h+j} (\chi_{\{1,2,...,\alpha_o\}}(k_l) \quad + \quad \chi_{\{\alpha_o+1,...,\alpha\}}(k_l) \chi_{E(\omega)}(e(k_1, ..., k_l))). \tag{11}$$

Proof: The multiple sum is a sum over the leaves of a homogeneous tree of finite height, while the product (over l) concerns the edges of a path leading from the root to each of the leaves of these trees. The factors of the products correspond to indicators of the events of the corresponding edges being open or closed. □

Since $X_j^{(h)}$ is a subtree of a Galton-Watson tree, it is clear that $\mathbb{E}[X_j^{(h)}] = z^{j+h+1}$, where

$$z := (1-p) \cdot \alpha_o \quad + \quad p \cdot \alpha = \alpha_o + p(\alpha - \alpha_o).$$

and z' is the corresponding primed parameter. This follows also by using (11) by evaluating the expected value at the leaves at the highest level, first.

In order to simplify the notation used in (10), let $X'_j(\omega') = X_j'^{(h)}(\omega')$ and $X_j(\omega) = X_j^{(h)}(\omega)$. We are employing Jensen's inequality in the following way: For any finite sequence $\{x_j\}_{j=1}^N$,

$$\frac{1}{N}\sum_{i=1}^{N}\frac{1}{x_i} \geq \frac{N}{\sum_{i=1}^{N} x_i}. \tag{12}$$

If $N = 2h+1$, then (12) applied to (10) gives

$$I_h \quad \leq \quad \frac{1}{(2h+1)^2}\sum_{j=-h}^{h}\frac{X'_h + X_h}{X'_{-j}X_j}. \tag{13}$$

Define $Y_j := X_j^{(h)}/z^{h+1+j}$, and $Y'_j := X_j'^{(h)}/z^{h-j}$. As is well known, Y_j is a martingale [10]. Therefore, $\mathbb{E}[Y_{j+1} - Y_j] = 0$, and $\mathbb{E}[X_h/X_j] = z^{h-j}\mathbb{E}[Y_h/Y_j] = z^{h-j}(1 + \mathbb{E}[(Y_h - Y_j)/Y_j]) = z^{h-j}(1 + \mathbb{E}[Y_h - Y_j]\mathbb{E}[1/Y_j]) = z^{h-j}$. So, due to the independence between trees, if $\mathbb{E}_\Omega$ is the expectation value obtained by integration (only) over Ω,

$$\mathbb{E}_\Omega\left[\frac{X_h}{X'_{-j}X_j}\right] \quad = \quad \frac{1}{X'_{-j}}\left(\, z^{h-j}\,\right) = \frac{1}{Y'_j}\,. \tag{14}$$

Similarly, denoting by $\mathbb{E}_{\Omega'}$ integration over Ω', we have $\mathbb{E}_{\Omega'}\left[\frac{X'_h}{X'_{-j}X_j}\right] = \frac{1}{Y_j}$. It follows,

$$\mathbb{E}_\Omega\left[\frac{X_h}{\sum X'_{-j}X_j}\right] \leq \frac{2}{(2h+1)^2}\sum_{j=-h}^{h}\frac{1}{Y'_j}, \qquad \mathbb{E}_{\Omega'}\left[\frac{X'_h}{\sum X'_{-j}X_j}\right] \leq \frac{2}{(2h+1)^2}\sum_{j=-h}^{h}\frac{1}{Y_j}, \tag{15}$$

for every $h \in \mathbb{N}$.

Since $\mathbb{E}_\Omega Y_j < \infty$, by the martingale convergence theorem, the sequence Y_j converges μ-almost surey. Moreover, under the assumption of non-extinction, the probability that $\lim Y_j = 0$ is zero, by the Kesten-Stigum theorem [16]. The same statement also holds for Ω and Y_j replaced by Ω' and Y_j'. This implies

$$\widehat{C}(\omega) := \sup_{j\in\mathbb{N}} \frac{1}{Y_j(\omega)} < \infty, \;\; \mu - \text{a.s.}, \qquad \text{and} \qquad \widehat{C}'(\omega') := \sup_{j\in\mathbb{N}} \frac{1}{Y_j'(\omega')} < \infty, \quad \mu' - \text{a.s..}$$

with $\widehat{C}(\omega), \widehat{C}'(\omega')$ independent of h. Using (15), this implies that

$$\mathbb{E}_\Omega \left[\frac{X_h}{\sum X'_{-j}(\omega')X_j} \right] \leq \frac{\widehat{C}'(\omega')}{(2h+1)}, \quad \mu' - \text{a.s.}$$

and

$$\mathbb{E}_{\Omega'} \left[\frac{X'_h}{\sum X'_{-j}X_j(\omega)} \right] \leq \frac{\widehat{C}(\omega)}{(2h+1)},, \quad \mu - \text{a.s..}$$

Therefore, $\bar{\mu}$-almost surely,

$$I_h(\omega,\omega') = O(h^{-1}), \qquad \text{as } h \to \infty.$$

Since $\bar{\mu}$-almost surely, $\{I_h\}_h$ converges to zero as $h \to \infty$, the sequence of sub-graphs $T_j'^{(h)}(\omega') \circ T_j^{(h)}(\omega)$ restricted to the connected component $C_{\bar{o}}(\bar{\omega})$ is a Følner sequence, $\bar{\mu}-a.s.$. Since it is a sequence of connected sub-graphs each containing the root $\bar{o} = \langle o', o\rangle$, the horocyclic product $T_j'(\omega') \circ T_j(\omega)$ restricted to the connected component $C_{\bar{o}}(\bar{\omega})$ is $\bar{\mu}$-a.s. strongly amenable. □

Remark: iv.) Theorem 2.4,ii. in [11] gives a comparable result for Bernoulli percolation on transitive graphs. In our model, the cases of either $\alpha_o = 0$ and $\alpha_o' = 0$ (both trees' edges all marked), or $\alpha_o = 0$ and $\alpha_o' = \alpha'$ (one tree being deterministic) imply that all the edges of $DL_{\alpha',\alpha}$ are percolative. However, this is different from Bernoulli percolation in that there are strong correlations between the openness of the edges $e' \circ e_1$ and $e' \circ e_2$.

3 Amenability of products of differing percolative trees

We now proceed to the situation where the 'anchor' in the assumptions is removed and instead of a Følner sequence of connected subsets with a root, only finiteness of the elements of this sequence is required. Recall that a graph $\tilde{G}$ is amenable iff $\inf |\partial_{\tilde{G}} C|/|C| = 0$, where the infimum is taken over all finite subsets of $V(\tilde{G})$. As before, $\tilde{G}|C$ will denote the subgraph of $\tilde{G}$ induced by the subset C of $V(\tilde{G})$. Once more, let $H(\bar{\omega}) = \langle V, \bar{E}(\bar{\omega})\rangle$ denote the horocyclic product $T'(\omega') \circ T(\omega)$.

For two vertices $u, v \in V(\mathcal{T})$ we will denote by $u \prec v$ the event that v is a vertex of a sub-tree of $T(\omega)$ with u being the root. If $u \prec v$ occurrs, this implies that $\mathbf{h}(u) < \mathbf{h}(v)$, and that u is an ancestor of v in the hierarchy induced by the Busemann-function $\mathbf{h}$. Similarly, for $u', v' \in V(\mathcal{T}')$, we define $u' \prec v'$ to mean $\mathbf{h}'(u') < \mathbf{h}'(v')$, and u' is an $'v'$ ancestor in the hierarchy induced by $\mathbf{h}'$.

Theorem 3.1. *Let $H(\bar{\omega}) = T'(\omega') \circ T(\omega)$ be a horocyclic product of two trees with offspring-measures which have non-empty support., i.e. let $\{\alpha'_o, ..., \alpha\} \cap \{\alpha_o, ..., \alpha\} \neq \emptyset$. Let $p', p \in (0, 1)$. Then $H(\bar{\omega})|C_{\bar{o}}(\bar{\omega})$ is almost surely amenable.*

Consider at first the horocyclic product of two finite percolation sub-trees with equal number of minimal offspring $\alpha'_o = \alpha_o$, maximal offspring $\alpha' = \alpha$, and equal height $N \in \mathbb{N}$. Choose $\bar{\omega} \in \Omega' \times \Omega$ such that

$$\deg(\bar{v}) = 2\alpha_o. \tag{16}$$

This is the event that all percolative edges are closed. By definition of $\bar{\mu}$ (see (8)), the probability that this occurs on both trees is $(1-p')^{2M_N}(1-p)^{2M_N} > 0$, where $M_N = \alpha \cdot (\alpha_o^{N+1} - 1)/(\alpha_o - 1)$, the factor after α being the number of vertices in a α_o-regular tree of height N. Likewise, the event of each vertex in either of two rooted sub-trees within a certain finite intervall of levels having the same number β of offspring has also positive probability. Due to the invariance of $\bar{\mu}$ under shifts accross different horocycles (=levels), any arbitrary number of such events occurrs also with positive probability. Moreover, due to the independence of the openness of edges between vertices on different horocycles, the measure is ergodic with respect to this shift.

Proof: (Theorem 3.1) Since $\alpha_o \geq 1$, the tree $T(\omega)$ is infinite. Choosing any sequence $(v_j)_{j\in\mathbb{N}}$ of vertices in $V(T(\omega))$ with $v_0 = o$, and $\mathbf{h}(v_n) = n$, we may consider the induced sequence of vertices $(\bar{v}_n)_{n\in\mathbb{N}}$ with $\bar{v}_n = \langle v'_n, v_n\rangle \in C_{\bar{o}}$, and $v'_0 = o' \in V(T'(\omega'))$. Note that $\mathbf{h}'(v'_n) = -n$, and that therefore $v'_{j+N} \prec v'_n$, for every positive $N \in \mathbb{N}$.

Let $Z_v(\omega)$ be the number of offspring of $v \in V(T(\omega))$, and $Z'_{v'}(\omega')$ the number of offspring of $v' \in V(T'(\omega'))$. For given $\beta \in \{\alpha'_o, ..., \alpha'\} \cap \{\alpha_o, ..., \alpha\}$, $N \in \mathbb{N}$, and $j \in \mathbb{N}$, the event

$$C_j^N := \left\{\langle\omega', \omega\rangle \in \bar{\Omega} \;:\; Z'_{u'}(\omega') = Z_u(\omega) = \beta \text{ for } \langle u', u\rangle \in C_{\bar{o}}(\bar{\omega}) \;\text{ s.t. }\; v_j \prec u,\;\; v'_{j+N} \prec u'\right\}$$

occurrs with positive probability $\bar{\mu}(C_j^N) > 0$.

The measure $\bar{\mu}$ is ergodic under the shift $\bar{v_k} \mapsto \bar{v}_{k+1}$ because of the independence of the openness of different edges (in particular of edges on different horocycles). Therefore, there is an $n \in \mathbb{N}$ for each $N \in \mathbb{N}$, such that C_n^N occurrs. Defining the random sequence $n.(\bar{w}) \in \mathbb{N}^{\mathbb{N}}$ by

$$n_N(\bar{\omega}) := \inf \left\{ k \in N \ : \ \bar{\omega} \in C_k^N \ \right\},$$

we identify a sequence of vertex sub-sets (called *tetraeder* in [2]):

$$V_N \{ \langle u', u \rangle \in C_{\bar{o}}(\bar{\omega}) \ : \ v_{n(N)} \prec u, \ \text{and} \ v'_{n(N)+N} \prec u' \}.$$

This sequence is a Følner-sequence, since the isoperimetric constant of the finite subgraph of $H(\bar{w})$ induced by $V_N(\bar{\omega})$ is given by

$$I_H(H(\bar{\omega})|V_N(\bar{\omega})) = \frac{|\partial_{H(\bar{\omega})} V_N(\bar{\omega})|}{|V_N(\bar{\omega})|} = \frac{2\beta^N}{\sum_{j=0}^N \beta^j \beta^{N-j}} = \frac{2}{N+1}.$$

□

Remarks: vi.) In this proof, for the construction of the Følner sequence it is important that with positive probability the graph locally contains arbitrarily large but finite subgraphs of the graphs induced by $C_o(\bar{\omega})$, which are (finite) symmetric horocyclic products.

vii.) The proof can be transferred to the situation in which dependent percolation prevails, however, with independence between the different trees and stationarity and stationary ergodicity with respect to the shift between the different horocycles (levels) within a single tree.

4 Non-amenability

One difficulty in bounding the anchored isoperimetric constant of subgraphs of horocyclic products is, that these subgraphs need not be horocyclic products, themselves. We will overcome this difficulty by removing additional edges from $H(\bar{\omega})$ and recognising the remaining graph to have a uniformly bounded isoperimetric ratio in a deterministic, non-amenable horocyclic product (compare with [21], [19]). Theorem 1.1 in [5] and Theorem 2.4,i. in [11] refer to Bernoulli percolation on a not necessarily transitive but locally finite graph. Again, this situation is realised in our model if $\alpha_o = 0$ and $\alpha'_o \in \{0, \alpha'\}$.

Using the idea, that the removal of edges may lead to non-amenable subgraphs, we formulate the following lemma. (These are two statements one with, the other without the parentheses.)

Lemma 4.1. *If, given a graph $G = \langle V, E\rangle$ of bounded degree, there is a subset of edges $E' \subset E$, such that for every finite, connected induced subgraph $G|C$ with $C \subset V$ (containing the root), the connected components of $G \setminus^{e} E' \mid C$ each have an isoperimetric ratio in $G \setminus^{e} E'$ uniformly bounded from below by $i_o > 0$, G is (weakly) non-amenable with (anchored) isoperimetric constant greater or equal to i_o.*

Proof: By lemma A3.3 of [11], since the graph G has bounded degree, it is sufficient to consider only *connected* subgraphs $G|C$ in the assumption (for non-amenability - for weak non-amenability, 'connected' requires no further justification). Let $\{C_j\}$ with $C_j \subset C$ be the finite set of disjoint subsets for which $(G \setminus^{e} E')|C = \cup_j (G \setminus^{e} E')|C_j = \sum_j G|C_j$. In other words, after taking away the edges E' of $G|C$, we are left with the disjoint subgraphs $G|C_j$. Since by assumption, each of them fulfills $|\partial_{G\setminus^{e} E'} C_j| \geq i_o |C_j|$ for some positive i_o (independent of C), it holds that

$$\frac{|\partial_G C|}{|C|} \geq \frac{|\partial_G C \setminus E'|}{|C|} = \frac{|\partial_{G\setminus^{e} E'} C|}{|C|} = \frac{1}{|C|} \sum_j |\partial_{G\setminus^{e} E'} C_j| \geq \frac{i_o}{|C|} \sum_j |C_j| = i_o. \quad \square$$

What if the smallest possible number of offspring of one of the trees (α_o) is larger than the largest number of offspring of the other tree (α')? We answer this question in the case of one tree being deterministic ($\alpha'_o = \alpha'$).

Theorem 4.2. *Let $\alpha'_o = \alpha' < \alpha_o < \alpha$. Then there is non-amenability,* for all *realisations of the random subgraphs $\mathcal{T}'_{\alpha'} \circ T_{\alpha_o,\alpha}(\omega)$ of $DL_{\alpha',\alpha}$.*

Proof: Note that when all the percolative edges E_p are removed ($p = 0$), the remaining graph is disconnected and all its connected components infinite and non-amenable. The isoperimetric ratio of any connected sub-graph induced by a finite subset of verticies $W \subset V$ fulfills

$$\frac{|\partial_H W|}{|W|} \geq \frac{|\partial_{H\setminus^{e} E_p} W|}{|W|},$$

which is of the form $(\sum_j |\partial_{H\setminus^e E_p} W_j|)/\sum_j |W_j|$, for a finite number n of subsets W_j of W, where the graph induced by each W_j is, by Lemma 1.1, the connected component resulting from removing the percolative edges E_p. Note that each of the corresponding isoperimetric ratios $I_j = |\partial_{H\setminus^e E_p} W_j|/|W_j|$ is uniformly bounded away from zero, since it is a subgraph of $\mathrm{DL}_{\alpha',\alpha_o}$, which is non-amenable, and by assumption $\alpha' < \alpha_o$. Given n ratios $|\partial W_j|/|W_j| \geq c$, uniformly bounded by $c > 0$, we have that $\sum_j |\partial W_j| / \sum_j |W_j| \geq c$. This means that *every* realisation of H is non-amenable. □

Remark: viii.) Note that a finite graph has vanishing isoperimetric constant.

5 Summary, Outlook and Acknowledgements

In this paper, it was proven that there are transitive graphs with certain independent percolation processes for which either strong amenability or amenability prevails, depending on the choice of the retention parameters. Two methods have been introduced to investigate amenability of percolative subgraphs: 1.) The expected isoperimetric ratio may lead to the existence of a Følner sequence, and 2.) the removal of edges may allow comparison with random graphs for which non-amenability has been proven.

Question: Is the range of strong amenability restricted to the assumption given in Theorem 2.1, or is there a non-trivial regime of the parameters p', p for anchored expansion (weak non-amenability) to prevail?

The question wether for some horocyclic products of trees drawn from the augmented Galton-Watson measure there is (strong) amenability together with simple random walk having positive speed is answered in a forthcoming paper of the author with V. Kaimanovich.

The question to what extent similar results hold for non-random periodic and quasi-periodic trees is investigated in a project by the author, D. Lenz, and I. Veselič.

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